Case $I_2(t)$: As I_2 is a continuous local martingale with $I_2(0) = 0$ it follows from the martingale property and that T_m reduces I_2 that $\mathbb{E}[I_2(t \wedge T_m)] = 0$.

Case $I_3(t)$: Using that $\varphi_n''(x) = \psi_n(|x|) \leq \frac{2}{n}\rho(|x|)^{-2}$ and the assumption $|\sigma(x) - \sigma(y)| \leq \rho(|x - y|)$ we get

$$\begin{aligned}\mathbb{E}[|I_3(t \wedge T_m)|] &= \tfrac{1}{2}\,\mathbb{E}\Big[\int_0^{t\wedge T_m} \varphi_n''(\Delta_s)(\sigma(X_s^1) - \sigma(X_s^2))^2\,\mathrm{d}s\Big] \\ &\leq \tfrac{1}{2}\,\mathbb{E}\Big[\int_0^{t\wedge T_m} \psi_n(|\Delta_s|)(\rho(|X_s^1 - X_s^2|)^2\,\mathrm{d}s\Big] \\ &\leq \tfrac{1}{n}\,\mathbb{E}\Big[\int_0^{t\wedge T_m} \rho(|\Delta_s|)^{-2}\rho(|\Delta_s|)^2\,\mathrm{d}s\Big] \\ &\leq \tfrac{t}{n}.\end{aligned}$$

Putting all the pieces together we see that

$$\begin{aligned}\mathbb{E}[\varphi_n(\Delta_{t\wedge T_m})] &= \mathbb{E}[I_1(t \wedge T_m)] + \mathbb{E}[I_2(t \wedge T_m)] + \mathbb{E}[I_3(t \wedge T_m)] \\ &\leq \mathbb{E}[|I_1(t \wedge T_m)|] + \mathbb{E}[|I_3(t \wedge T_m)|] \\ &\leq K_m \int_0^t \mathbb{E}[|\Delta_{s\wedge T_m}|]\,\mathrm{d}s + \tfrac{t}{n}.\end{aligned}$$

Setting $g(t) = \mathbb{E}[|\Delta_{t\wedge T_m}|]$ we find using Fatou's Lemma that

$$\begin{aligned}g(t) = \mathbb{E}[|\Delta_{T_m \wedge t}|] &\leq \liminf_{n\to\infty} \mathbb{E}[\varphi_n(\Delta_{t\wedge T_m})] \\ &\leq K_m \int_0^t \mathbb{E}[|\Delta_{s\wedge T_m}|]\,\mathrm{d}s = K_m \int_0^t g(s)\,\mathrm{d}s.\end{aligned}$$

We have that $g(t)$ is non-negative and finite since

$$g(t) = \mathbb{E}[|\Delta_{T_m\wedge t}|] = \mathbb{E}[|X^1_{T_m\wedge t} - X^2_{T_m\wedge t}|] \leq 2m < \infty.$$

As g is also continuous and satisfies $g(t) \leq K_m \int_0^t g(s)\,\mathrm{d}s$ it follows from Grönwall's Inequality that $g(t) = 0$ for all $t \geq 0$. This implies that $\Delta_{t\wedge T_m} = 0$ almost surely for $t \geq 0$, and by letting m go to infinity it follows that $\Delta_t \simeq 0$, that is, $X_t^1 \simeq X_t^2$. □

10.26 · Remarks.

(A) A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **Hölder continuous of order $\alpha \in (0, 1]$** if there exists a $K \geq 0$ such that

$$|f(x) - f(y)| \leq |x - y|^\alpha \qquad \text{for all } x, y \in \mathbb{R}.$$

(B) If $\alpha = 1$ then f is Lipschitz continuous with Lipschitz constant 1.

(C) If σ is Hölder continuous of order $\alpha \geq \frac{1}{2}$ then (i) in Theorem 10.25 is satisfied with $\rho(x) = x^\alpha$, as

$$\int_0^\varepsilon \rho(u)^{-2}\,\mathrm{d}u = \int_0^\varepsilon u^{-2\alpha}\,\mathrm{d}u = \infty \qquad \text{for every } \varepsilon > 0.$$

(D) The function $x \mapsto |x|^\alpha$ for $\alpha \geq \frac{1}{2}$ is Hölder continuous of order α.

(E) We cannot put the same mild restriction on b. For example if $\alpha < 1$ and $\sigma = 0$ and $b(x) = |x|^\alpha \wedge 1$ then it is Hölder continuous of order α as we see, using Exercise 10.2, that

$$|b(x) - b(y)| \leq \big||x|^\alpha - |y|^\alpha\big| \wedge 1 \leq |x - y|^\alpha.$$

By Exercise 10.4 pathwise uniqueness does not hold for the SDE(§). ◇

10.3 · Weak Solutions

Consider the space $(C, \mathcal{B}(C))$, that is, the space of continuous functions $f : \mathbb{R}_+ \to \mathbb{R}^d$ equipped with the Borel-algebra $\mathcal{B}(C)$. Let (X_t) be a continuous process, we show in Exercise 9.4 that the mapping $\Phi_X : \Omega \to C$ given by $\omega \mapsto X_\cdot(\omega)$ is $\mathcal{F}$-$\mathcal{B}(C)$-measurable. This means the distribution of the process (X_t), that is, $\mathbb{P}_X = \mathbb{P} \circ \Phi_X^{-1}$ is a well-defined measure on $(C, \mathcal{B}(C))$. Note that $\mathbb{P}_X$ is decided by the values on the intersection stable generating set $A = \{\xi_{t_1}^{-1}(A_1) \cap \cdots \cap \xi_{t_n}^{-1}(A_n) \mid n \in \mathbb{N}, A_i \in \mathcal{B}(\mathbb{R}^d), 0 \le t_1 \le \cdots \le t_n\}$ where $\xi_t : C \to \mathbb{R}^d$ is the projection $\xi_t(\omega) = \omega(t)$. For $n \in \mathbb{N}, A_i \in \mathcal{B}(\mathbb{R}^d), 0 \le t_1 < \cdots < t_n$ we see that

$$\mathbb{P}_X(\xi_{t_1}^{-1}(A_1) \cap \cdots \cap \xi_{t_n}^{-1}(A_n)) = \mathbb{P}((\xi_{t_1} \circ \Phi_X)^{-1}(A_1) \cap \cdots \cap (\xi_{t_n} \circ \Phi_X)^{-1}(A_n)) \tag{10.13}$$

$$= \mathbb{P}(X_{t_1} \in A_1, \dots, X_{t_n} \in A_n) \tag{10.14}$$

that is, $\mathbb{P}_X$ is characterized by the finite-dimensional distributions of $(X_{t_1}, \dots, X_{t_n})$ for $0 \le t_1 < \cdots < t_n$ for $n \in \mathbb{N}$. In particular one sees that all standard Brownian motions have the same distribution. One refers to this measure as the Wiener measure on the path space. Note that (ξ_t) is a standard Brownian motion when $(C, \mathcal{B}(C))$ is equipped with the Wiener measure and the filtration $\mathcal{F}_t = \sigma(\xi_s \mid s \le t)^*$. Furthermore we observe that if two continuous processes are modifications then they have identical distribution. We have the following definition.

10.27 · Definition. We say that there is **uniqueness in distribution** in the SDE(§) if for all $x \in \mathbb{R}^d$ and all solutions to the SDE(§) $(X_t, B_t, \mathcal{F}_t)$ on $(\Omega, \mathcal{F}, \mathbb{P})$ and $(X'_t, B'_t, \mathcal{F}'_t)$ on $(\Omega', \mathcal{F}', \mathbb{P}')$ such that $\mathbb{P}(X_0 = x) = \mathbb{P}'(X'_0 = x) = 1$ we have $\mathbb{P}_X = \mathbb{P}'_{X'}$.

10.28 · Example. This is an example where we show that pathwise uniqueness does not hold but uniqueness in distribution does. Let $d = m = 1$, $b = 0$ and $\sigma = \text{sign}$. We have shown in Example 10.10 that pathwise uniqueness does not hold in this case. Let $x \in \mathbb{R}$ and $X_t = x + \int_0^t \text{sign}(X_s)\,\mathrm{d}B_s$ and set $Y_t = X_t - x = \int_0^t \text{sign}(X_s)\,\mathrm{d}B_s$. Then (Y_t) is a continuous local martingale with $Y_0 = 0$ and

$$\langle Y \rangle_t \simeq \int_0^t \text{sign}(X_s)^2\,\mathrm{d}s = \int_0^t 1\,\mathrm{d}s = t, \qquad (t \ge 0)$$

implying that (Y_t) is a Brownian motion by Lévy's Theorem. If $(X'_t, B'_t, \mathcal{F}'_t)$ is another solution, then defining $Y'_t = X'_t - x$ we obtain

$$\mathbb{P}'_{X'} = \mathbb{P}'_{x+Y'} = \mathbb{P}_{x+Y} = \mathbb{P}_X,$$

as (Y'_t) is a Brownian motion and the distribution of a Brownian motion is unique, cf. Lemma A.27(i). ○

One might wonder about the relation between uniqueness in distribution and pathwise uniqueness of the SDE(§), but uniqueness in distribution concerns solutions defined on different probability spaces and as such it is not immediately clear that pathwise uniqueness is actually the strongest, but this is proven in the next theorem and its proof is based on regular conditional distribution and distributions of processes on path spaces. The reader is therefore encouraged to consult Appendix A.4 and A.6.

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* For more information look at Section A.4, specifically Lemma A.27.